Regular representations of finite groups via hypergraphs

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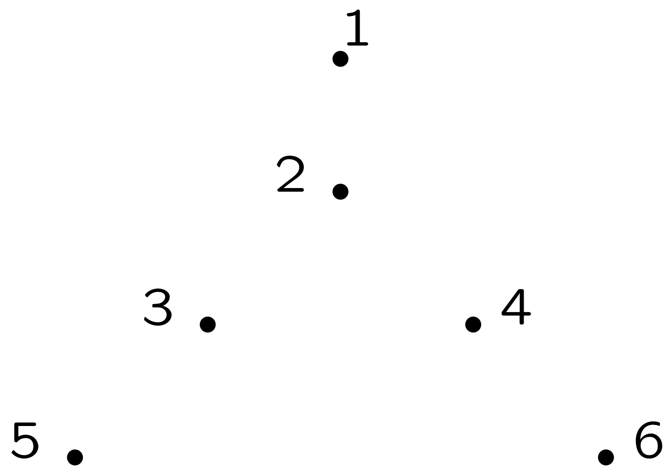
Theorem 1 (Cayley) *Every (finite) group* $(G, \cdot)$ *is isomorphic to a permutation group*

$$G_L \leq Sym_G$$

$$G_L = \{\sigma_a \mid \forall a \in G \;\; \sigma_a(b) = a \cdot b\}$$

G_L *is called the* **left regular representation** *of* G.

Example: $\mathcal{D}_3 = \{1_{\mathcal{D}_3}, r, r^2, s, sr, sr^2\}$



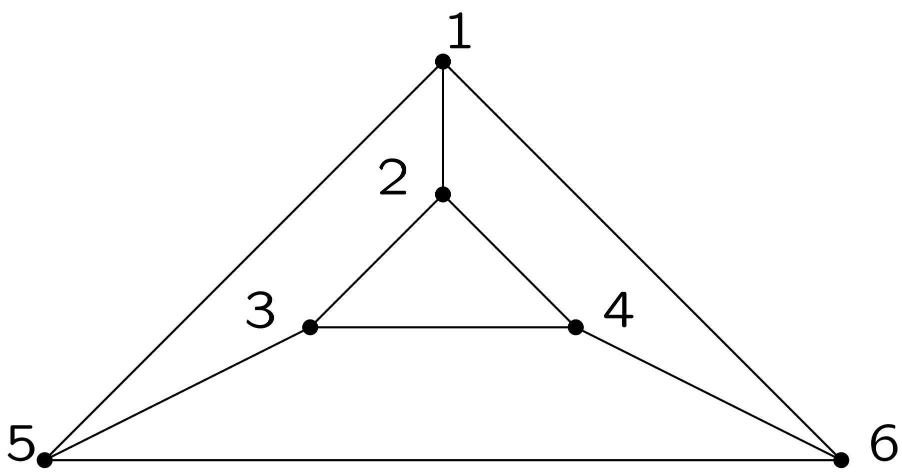
$1_{\mathcal{D}_3} \mapsto 1$, $s \mapsto 2$, $sr^2 \mapsto 3$, $sr \mapsto 4$, $r \mapsto 5$, $r^2 \mapsto 6$

$$\sigma_r = (1, 5, 6)(2, 3, 4)$$

Definition 1 *Let* $\Gamma = (V, E)$ *be a graph. The* **(full) automorphism group** *of* Γ, $Aut\Gamma$, *is the subgroup of* Sym_V *of all permutations* σ *that preserve the structure of* Γ, *i.e.,*

$$\sigma(u)\sigma(v) \quad \textit{iff} \quad uv$$

Example: Γ



$$\mathcal{D}_3 < Aut\Gamma$$

Question 1 *Is there a graph* Γ *such that* $Aut\Gamma$ *is exactly* $(\mathcal{D}_3)_L$*?*

Definition 2 *A graph* Γ *is a* **graphical regular representation** *for a group* G *if* $Aut\Gamma = G_L$*.*

Example: K_n is the GRR for Sym_n

Question 2 *Which finite groups allow for a GRR?*

Definition 3 *Given a (finite) group* G *and a set* X *of elements of* G*,* $X \subset G$*, closed under taking inverses,* $X = X^{-1}$*, and not containing* 1_G*, the* **Cayley graph** $\Gamma = C(G, X)$ *is the graph* (G, E) *where* $E = \{\ \{g, g \cdot x\} \mid g \in G, x \in X\}$*.*

Lemma 1 (Sabidussi) *A graph* $\Gamma = (V, E)$ *admits a regular subgroup* G *of the full automorphism group* $Aut\Gamma$ *if and only if* Γ *is a Cayley graph for* G*.*

Lemma 2 *Let G be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group or G is isomorphic to one of the 13 groups :* $\mathcal{Z}_2^2$, $\mathcal{Z}_2^3$, $\mathcal{Z}_2^4$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{A}_4$, $\mathcal{Q} \times \mathcal{Z}_3$, $\mathcal{Q} \times \mathcal{Z}_4$,
$\left\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \right\rangle$,
$\left\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5 \right\rangle$,
$\left\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \right\rangle$
$\left\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = a \right.$

Note that all the "exceptional" groups are of order ≤ 32.

Conjecture 1 *Almost all Cayley graphs are GRR's.*

Definition 4 *An* **incidence structure** *on a set* V *is any ordered pair* $\mathcal{I} = (V, \mathcal{B})$*, where* $\mathcal{B}$ *is a family of subsets of* V*,* $\mathcal{B} \subseteq \mathcal{P}(V)$*.*

The **automorphism group** $Aut\mathcal{I}$ *is the set of all permutations* σ *preserving the structure of* $\mathcal{I}$*, i.e.,* $\sigma(A) \in \mathcal{B}$ *iff* $A \in \mathcal{B}$*.*

Definition 5 *A* k-**hypergraph** *on a set* V *is any ordered pair* $\mathcal{I} = (V, \mathcal{B})$*, where* $\mathcal{B}$ *is a family of* k*- subsets of* V*,* $\mathcal{B} \subseteq \mathcal{P}_k(V)$*.*

Lemma 3 *A cyclic group* $\mathcal{Z}_i$ *admits a regular representation on an incidence structure if and only if* $i \neq 3, 4, 5$*. In fact, the incidence structure can be chosen to be a 3-hypergraph.*

Proof. Let $i \geq 6$, and take $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, $\mathcal{B}_1 = \{\ \{j, j+1, j+2\} \mid 1 \leq j \leq i\}$ and $\mathcal{B}_2 = \{\ \{j, j+1, j+3\} \mid 1 \leq j \leq i\}$ (with the addition modulo i). Then $Aut(\mathcal{Z}_i, \mathcal{B}) = \mathcal{Z}_i$, for all $i \geq 6$.

Lemma 4 *Let G be a generalized dihedral group. Then G can be represented as a regular full automorphism group of some combinatorial structure if and only if $G \neq \mathcal{Z}_2^2$.*

Lemma 5 *Let G be a finite group that admits an irreducible generating set X of size at least 3. Then G is regularly representable as the full automorphism group of some incidence structure $(G, \mathcal{B})$.*

Theorem 2 *A finite group G can be represented as a regular full automorphism group of some incidence structure if and only if G is not one of the groups $\mathcal{Z}_3$, $\mathcal{Z}_4$, $\mathcal{Z}_5$ or $\mathcal{Z}_2^2$.*

For the rest of the talk, we shall focus on representing finite groups as regular automorphism groups of hypergraphs.

Lemma 6 (Babai) *The finite group G admits a DRR if and only if G is neither the quaternion group $\mathcal{Q}$ nor any of $\mathcal{Z}_2^2$, $\mathcal{Z}_2^3$, $\mathcal{Z}_2^4$, $\mathcal{Z}_3^2$.*

Definition 6 *Let G be a (finite) group, and let $X_1, X_2, \ldots, X_{k-1}$ be subsets of G that do not contain the identity 1_G. The C_k-*hypergraph *$C_k(G; X_1, X_2, \ldots, X_{k-1})$ is the incidence structure $(G, \mathcal{B})$ with $\mathcal{B}$ being the set of all k-subsets of the form*

$$\{g, gx_1, gx_1x_2, \ \ldots, \ gx_1x_2\ldots x_{k-1}\},$$

$g \in G$, and $x_i \in X_i$ for $1 \leq i \leq k-1$.

Note that we strictly require that the blocks have exactly k vertices in order to be included, i.e., all the vertices $g, gx_1, gx_1x_2, \ \ldots, \ gx_1x_2\ldots x_{k-1}$ must be different.

Since graph automorphisms preserve k-arcs,

$$Aut(C(G,X)) \leq Aut(C_k(G;X,X,\ldots,X)).$$

Lemma 7 *Let $C(G,X)$ be a Cayley graph of girth $g > 2k-2$, $k \geq 2$, and valence $|X| > k-1$. Then $Aut(C(G,X)) = Aut(C_k(G;X,X,\ldots,X))$.*

Corollary 1 *Let G be a finite group that admits a GRR of girth $g > 2m-2$ and valence r. Then G can be regularly represented as the full automorphism group of some k-hypergraph for all $2 \leq k \leq$ min $\{m, r-1\}$.*

Lemma 8 *Let G be a finite group, $X_1, X_2, \ldots X_{k-1}$ be symmetric subsets of G not containing 1_G, and suppose that all the* reduced *words $x_1 x_2 \ldots x_l$, $x_i \in X_i$, $1 \leq l \leq k-1$, represent different elements of G. If $|X_i| > k-1$, for all $1 \leq i \leq k-1$, then $Aut(C_k(G;X_1,X_2,\ldots,X_{k-1})) \leq Aut(C(G,X_1))$.*

Corollary 2 *Let G be a finite group that admits a GRR $C(G, X_1)$, $Aut(C(G, X_1)) = G_L$. If there exist symmetric subsets $X_2, X_3, \ldots, X_{k-1}$ of G not containing 1_G, and such that all the* reduced *words $x_1 x_2 \ldots x_l$, $x_i \in X_i$, $1 \leq l \leq k-1$, represent different elements of G and $|X_i| > k-1$, for all $1 \leq i \leq k-1$, then $Aut(C_k(G; X_1, X_2, \ldots, X_{k-}$ G_L, and G admits a regular representation as the full automorphism group of a k-hypergraph.*

Lemma 9 *Let G be a finite abelian group that contains a cyclic subgroup of order at least 6 or a finite generalized dicyclic group with a normal abelian subgroup A of index 2 that contains a cyclic subgroup of order at least 6. Then G can be regularly represented as the full automorphism group of some 3-hypergraph on G.*

Lemma 10 *Let G be a finite group that admits an irreducible generating set X of size at least 4. Then G admits a regular representation as the full automorphism group of some 3-hypergraph.*

It follows that

- the only cyclic abelian groups that do not admit a representation through a 3-hypergraph are the groups

$$\mathcal{Z}_3,\ \mathcal{Z}_4,\ \text{ and } \mathcal{Z}_5$$

- the non-cyclic finite abelian groups not covered by the above lemmas are

$$Z_2^2, \mathcal{Z}_2^3, \mathcal{Z}_4 \times \mathcal{Z}_2,\ \mathcal{Z}_4 \times \mathcal{Z}_2^2,\ \mathcal{Z}_4^2,\ \mathcal{Z}_4^3,\ \mathcal{Z}_3^2,\ \mathcal{Z}_3^3,\ \mathcal{Z}_5^2,$$

$$\text{and } \mathcal{Z}_5^3$$

- the non-abelian groups not covered by the above lemmas are

$$\mathcal{D}_3,\ \mathcal{D}_4,\ \mathcal{D}_5,\ \mathcal{D}_4 \times \mathcal{Z}_2,\ \mathcal{D}_4 \times \mathcal{Z}_4,\ \mathcal{D}_3 \times \mathcal{Z}_3,$$

$$\text{and } \mathcal{D}_5 \times \mathcal{Z}_5$$

Theorem 3 *Let G be a finite group that does not admit a GRR. If G is not one of the above groups or one of 7 exceptional cases from the classification of GRR's, then G admits a regular representation as the full automorphism group of some 3-hypergraph.*

Conjecture 2 *If G is a finite group that admits a GRR, than G can be regularly represented as the full automorphism group of a k-hypergraph for all $2 \leq k \leq |G| - 2$.*

Conjecture 3 *If G is a finite group that does not admit a GRR and is not one of the groups excluded in the above Theorem, then G can be regularly represented as the full automorphism group of a k-hypergraph for all $3 \leq k \leq |G| - 3$.*