Regular representations of finite groups via hypergraphs

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Theorem 1 (Cayley) Every (finite) group $(G, \cdot)$ is isomorphic to a permutation group

$$G_L \leq \text{Sym}_G$$

$$G_L = \{ \sigma_a \mid \forall a \in G \ \sigma_a(b) = a \cdot b \}$$

$G_L$ is called the left regular representation of $G$.

Example: $D_3 = \{1_{D_3}, r, r^2, s, sr, sr^2\}$

\[
\begin{array}{cccc}
& 1 & & \\
1 & & & \\
& 2 & & \\
& & 3 & 4 \\
& & & 5 & 6 \\
1_{D_3} & \mapsto & 1, & s & \mapsto 2, & sr^2 & \mapsto 3, & sr & \mapsto 4, & r & \mapsto 5, & r^2 & \mapsto 6 \\
\sigma_r = (1, 5, 6)(2, 3, 4)
\end{array}
\]
Definition 1 Let $\Gamma = (V, E)$ be a graph. The (full) automorphism group of $\Gamma$, $\text{Aut}\Gamma$, is the subgroup of $\text{Sym}_V$ of all permutations $\sigma$ that preserve the structure of $\Gamma$, i.e.,

$$\sigma(u)\sigma(v) \iff uv$$

Example: $\Gamma$ 

\[
\begin{array}{c}
\text{D}_3 < \text{Aut}\Gamma
\end{array}
\]
Question 1 Is there a graph $\Gamma$ such that $\text{Aut}\Gamma$ is exactly $(D_3)_L$?

Definition 2 A graph $\Gamma$ is a graphical regular representation for a group $G$ if $\text{Aut}\Gamma = G_L$.

Example: $K_n$ is the GRR for $\text{Sym}_n$.

Question 2 Which finite groups allow for a GRR?

Definition 3 Given a (finite) group $G$ and a set $X$ of elements of $G$, $X \subset G$, closed under taking inverses, $X = X^{-1}$, and not containing $1_G$, the Cayley graph $\Gamma = C(G, X)$ is the graph $(G, E)$ where $E = \{ \{g, g \cdot x\} \mid g \in G, x \in X\}$.

Lemma 1 (Sabidussi) A graph $\Gamma = (V, E)$ admits a regular subgroup $G$ of the full automorphism group $\text{Aut}\Gamma$ if and only if $\Gamma$ is a Cayley graph for $G$. 
Lemma 2  Let $G$ be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then $G$ is an abelian group of exponent greater than 2 or $G$ is a generalized dicyclic group or $G$ is isomorphic to one of the 13 groups: $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $D_3$, $D_4$, $D_5$, $A_4$, $Q \times \mathbb{Z}_3$, $Q \times \mathbb{Z}_4$,
\begin{align*}
\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle, \\
\langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^5 \rangle, \\
\langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, (ac)^2 = (bc)^2 = 1 \rangle, \\
\langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, bc = cb, b^{-1}ab = a \rangle.
\end{align*}

Note that all the “exceptional” groups are of order \( \leq 32 \).

Conjecture 1  Almost all Cayley graphs are GRR’s.
Definition 4 An incidence structure on a set \( V \) is any ordered pair \( \mathcal{I} = (V, \mathcal{B}) \), where \( \mathcal{B} \) is a family of subsets of \( V \), \( \mathcal{B} \subseteq \mathcal{P}(V) \).

The automorphism group \( \text{Aut}\mathcal{I} \) is the set of all permutations \( \sigma \) preserving the structure of \( \mathcal{I} \), i.e., \( \sigma(A) \in \mathcal{B} \) iff \( A \in \mathcal{B} \).

Definition 5 A \( k \)-hypergraph on a set \( V \) is any ordered pair \( \mathcal{I} = (V, \mathcal{B}) \), where \( \mathcal{B} \) is a family of \( k \)-subsets of \( V \), \( \mathcal{B} \subseteq \mathcal{P}_k(V) \).

Lemma 3 A cyclic group \( \mathbb{Z}_i \) admits a regular representation on an incidence structure if and only if \( i \neq 3, 4, 5 \). In fact, the incidence structure can be chosen to be a 3-hypergraph.

Proof. Let \( i \geq 6 \), and take \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \), \( \mathcal{B}_1 = \{ \{ j, j + 1, j + 2 \} \mid 1 \leq j \leq i \} \) and \( \mathcal{B}_2 = \{ \{ j, j + 1, j + 3 \} \mid 1 \leq j \leq i \} \) (with the addition modulo \( i \)). Then \( \text{Aut}(\mathbb{Z}_i, \mathcal{B}) = \mathbb{Z}_i \), for all \( i \geq 6 \).
Lemma 4  Let $G$ be a generalized dihedral group. Then $G$ can be represented as a regular full automorphism group of some combinatorial structure if and only if $G \neq \mathbb{Z}_2^2$.

Lemma 5  Let $G$ be a finite group that admits an irreducible generating set $X$ of size at least 3. Then $G$ is regularly representable as the full automorphism group of some incidence structure $(G, B)$.

Theorem 2  A finite group $G$ can be represented as a regular full automorphism group of some incidence structure if and only if $G$ is not one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2^2$. 
For the rest of the talk, we shall focus on representing finite groups as regular automorphism groups of hypergraphs.

**Lemma 6 (Babai)** The finite group $G$ admits a DRR if and only if $G$ is neither the quaternion group $Q$ nor any of $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_3^2$, $\mathbb{Z}_4^2$.

**Definition 6** Let $G$ be a (finite) group, and let $X_1, X_2, \ldots, X_{k-1}$ be subsets of $G$ that do not contain the identity $1_G$. The $C_k$-hypergraph $C_k(G; X_1, X_2, \ldots, X_{k-1})$ is the incidence structure $(G, B)$ with $B$ being the set of all $k$-subsets of the form

$$\{g, gx_1, gx_1x_2, \ldots, gx_1x_2\ldots x_{k-1}\},$$

$g \in G$, and $x_i \in X_i$ for $1 \leq i \leq k - 1$.

Note that we strictly require that the blocks have exactly $k$ vertices in order to be included, i.e., all the vertices $g, gx_1, gx_1x_2, \ldots, gx_1x_2\ldots x_{k-1}$ must be different.
Since graph automorphisms preserve $k$-arcs,

$$\text{Aut}(C(G, X)) \leq \text{Aut}(C_k(G; X, X, \ldots, X)).$$

**Lemma 7** Let $C(G, X)$ be a Cayley graph of girth $g > 2k - 2$, $k \geq 2$, and valence $|X| > k - 1$. Then $\text{Aut}(C(G, X)) = \text{Aut}(C_k(G; X, X, \ldots, X)).$

**Corollary 1** Let $G$ be a finite group that admits a GRR of girth $g > 2m - 2$ and valence $r$. Then $G$ can be regularly represented as the full automorphism group of some $k$-hypergraph for all $2 \leq k \leq \min \{m, r - 1\}$.

**Lemma 8** Let $G$ be a finite group, $X_1, X_2, \ldots X_{k-1}$ be symmetric subsets of $G$ not containing $1_G$, and suppose that all the reduced words $x_1 x_2 \ldots x_l$, $x_i \in X_i$, $1 \leq l \leq k - 1$, represent different elements of $G$. If $|X_i| > k - 1$, for all $1 \leq i \leq k - 1$, then $\text{Aut}(C_k(G; X_1, X_2, \ldots, X_{k-1})) \leq \text{Aut}(C(G, X_1)).$
Corollary 2 Let $G$ be a finite group that admits a GRR $C(G, X_1)$, $\text{Aut}(C(G, X_1)) = G_L$. If there exist symmetric subsets $X_2, X_3, \ldots, X_{k-1}$ of $G$ not containing $1_G$, and such that all the reduced words $x_1x_2\ldots x_l$, $x_i \in X_i$, $1 \leq l \leq k-1$, represent different elements of $G$ and $|X_i| > k-1$, for all $1 \leq i \leq k-1$, then $\text{Aut}(C_k(G; X_1, X_2, \ldots, X_{k-1})) = G_L$, and $G$ admits a regular representation as the full automorphism group of a $k$-hypergraph.

Lemma 9 Let $G$ be a finite abelian group that contains a cyclic subgroup of order at least 6 or a finite generalized dicyclic group with a normal abelian subgroup $A$ of index 2 that contains a cyclic subgroup of order at least 6. Then $G$ can be regularly represented as the full automorphism group of some 3-hypergraph on $G$.

Lemma 10 Let $G$ be a finite group that admits an irreducible generating set $X$ of size at least 4. Then $G$ admits a regular representation as the full automorphism group of some 3-hypergraph.
It follows that

- the only cyclic abelian groups that do not admit a representation through a 3-hypergraph are the groups
  \[ \mathbb{Z}_3, \mathbb{Z}_4, \text{ and } \mathbb{Z}_5 \]

- the non-cyclic finite abelian groups not covered by the above lemmas are
  \[ \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2^2, \mathbb{Z}_4^2, \mathbb{Z}_4^3, \mathbb{Z}_3^2, \mathbb{Z}_3^3, \mathbb{Z}_5^2, \text{ and } \mathbb{Z}_5^3 \]

- the non-abelian groups not covered by the above lemmas are
  \[ \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_4 \times \mathbb{Z}_2, \mathcal{D}_4 \times \mathbb{Z}_4, \mathcal{D}_3 \times \mathbb{Z}_3, \text{ and } \mathcal{D}_5 \times \mathbb{Z}_5 \]
Theorem 3 Let $G$ be a finite group that does not admit a GRR. If $G$ is not one of the above groups or one of 7 exceptional cases from the classification of GRR’s, then $G$ admits a regular representation as the full automorphism group of some 3-hypergraph.

Conjecture 2 If $G$ is a finite group that admits a GRR, than $G$ can be regularly represented as the full automorphism group of a $k$-hypergraph for all $2 \leq k \leq |G| − 2$.

Conjecture 3 If $G$ is a finite group that does not admit a GRR and is not one of the groups excluded in the above Theorem, then $G$ can be regularly represented as the full automorphism group of a $k$-hypergraph for all $3 \leq k \leq |G| − 3$. 

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