We continued discussing the isomorphisms of groups.

First we listed some necessary conditions for two groups to be isomorphic, e.g., they have to be of the same order, they have to have the same distribution of orders of their elements, both have to be commutative or non-commutative at the same time.

Then we attempted to construct all the isomorphisms between the group $\mathbb{Z}_4$ and the group of permutations $\{((), (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2))\}$. We checked the orders and figured that $() \mapsto 0$, and $(1, 3)(2, 4) \mapsto 2$. Then we said that we have options for the image of $(1, 2, 3, 4)$ – either 1 or 3, and each choice can be completed into an isomorphism. We concluded that there are 2 isomorphisms between the two groups.

Ten we defined the group of automorphisms of a group, and discussed the following theorem:

Let $G$ and $H$ be isomorphic groups. Then the number of isomorphisms from $G$ to $H$ is equal to the order of the automorphism group of $G$ which is also equal to the order of the automorphism group for $H$.

You are welcome to prove this theorem.

Finally, we discussed what it means to classify all finite groups up to isomorphism.

We have obtained the following lists:

- groups of order 1: unique, $\{1_G\}$
- groups of order 2: unique, $\mathbb{Z}_2$
- groups of order 3: unique, $\mathbb{Z}_3$
- groups of order 4: exactly 2, $\mathbb{Z}_4$, and the Klein 4-group, $V_4 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$
- groups of order 5: unique, $\mathbb{Z}_5$, because 5 is a prime
- groups of order 6: exactly 2, $\mathbb{Z}_6$ and $S_3$
- ...
- groups of order 8: at least $\mathbb{Z}_8$, $D_4$, $Q_8$, $\mathbb{Z}_2^4$
- ...

Note: if $n$ is a prime, then there is exactly one group of order $n$, namely $\mathbb{Z}_n$. 